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LETTER TO THE EDITOR

The q-harmonic oscillator and an analogue of the Charlier polynomials

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Abstract. A model of a q-harmonic oscillator based on q-Charlier polynomials of Al-Salam and Carlitz is discussed. A simple explicit realization of q-creation and q-annihilation operators, q-coherent states and an analogue of the Fourier transformation are found. A connection of the kernel of this transform with biorthogonal rational functions is observed.

Models of q-harmonic oscillators are being developed in connection with quantum groups and their various applications (see, for example, [1–5]). The q-analogues of boson operators have been introduced explicitly in [1, 3, 5], where the corresponding wavefunctions were found in terms of the Rogers–Szegö polynomials [6], in terms of the continuous q-Hermite polynomials of Rogers [7, 8] and the Stieltjes–Wigert polynomials [9, 10], respectively. Here we introduce one more explicit realization of q-creation and q-annihilation operators with the aid of q-Charlier polynomials of Al-Salam and Carlitz [11].

The q-orthogonal polynomials $V_n^a(x;q)$ studied by Al-Salam and Carlitz may be considered as a *q-version of the Charlier polynomials* $c_n^\mu(s)$ (see, for example, [12, 13]). To emphasize this analogy we use the notation $v_n^\mu(x;q)$ for the Al-Salam and Carlitz polynomials. In our notation they can be defined by the three-term recurrence relation

$$\mu q^{-n-1} v_{n+1}^\mu(x;q) + (1-q^n)\, q^{-n} v_{n-1}^\mu(x;q) = \left((\mu+q)q^{-n-1} - x\right) v_n^\mu(x;q) \tag{1}$$

with $v_0^\mu(x;q) = 1$, $v_1^\mu(x;q) = \mu^{-1}(\mu + q - qx)$. These polynomials are orthogonal:

$$\sum_{s=0}^{\infty} v_m^\mu(q^{-s};q)\, v_n^\mu(q^{-s};q)\, \rho(s) q^{-s} = \frac{(q;q)_n}{\mu^n}\,\delta_{mn} \tag{2}$$

with respect to a positive measure

$$\rho(s) = (\mu;q)_\infty \frac{\mu^s q^{s^2}}{(q,\mu;q)_s} \qquad 0 < \mu, q < 1 \tag{3}$$

where the usual notations (see [14]) are

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \qquad (a,b;q)_n = (a;q)_n(b;q)_n \qquad (a;q)_\infty = \lim_{n\to\infty}(a;q)_n . \tag{4}$$

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The weight function (3) is a solution of the Pearson equation $\Delta(\sigma\rho) = \rho\tau\nabla x_1$ (here $\Delta f(s) = \nabla f(s+1) = f(s+1) - f(s)$ and $x_1(s) = x\left(s+\frac{1}{2}\right)$; for details, see [13, 15]) with $x(s) = q^{-s}$, $\sigma(s) = (1-q^{-s})(\mu - q^{1-s})$ and $\sigma(s) + \tau(s)\nabla x_1(s) = \mu$. The explicit form of the polynomials $v_n^{\mu}(x; q)$ is

$$v_n^{\mu}(x;q) = {}_2\varphi_0(q^{-n}, x; -; q, q^{1+n}/\mu),\ x = q^{-s}\,. \tag{5}$$

For the definition of the basic hypergeometric function ${}_2\varphi_0$, see [14]. In the limit $q \to 1$ it easy to obtain from (1) or (5)

$$\lim_{q\to 1} v_n^{(1-q)\mu}(q^{-s}; q) = {}_2F_0(-n, -s; -; -1/\mu) = c_n^{\mu}(s)\,. \tag{6}$$

This justifies our notation for the Al-Salam and Carlitz polynomials.

The polynomials $v_n^{\mu}(x; q)$ give us the possibility to introduce a new model of a q-oscillator. We can define a *q-version of the wavefunctions* of harmonic oscillator as

$$\psi_n(s) = d_n^{-1} q^{-s/2} \rho^{1/2}(s)\, v_n^{\mu}(q^{-s}; q) \tag{7}$$

where $d_n^2 = (q;q)_n/\mu^n$. These q-wavefunctions satisfy the orthogonality relation

$$\sum_{s=0}^{\infty} \psi_n(s)\, \psi_m(s) = \delta_{nm}\,. \tag{8}$$

The *q-annihilation* a and *q-creation* a^+ operators have the following explicit form:

$$\begin{aligned} a &= (1-q)^{-1/2}\left[\mu^{1/2}q^s - \sqrt{(1-q^{s+1})(1-\mu q^s)}\,\mathrm{e}^{\partial_s}\right] \\ a^+ &= (1-q)^{-1/2}\left[\mu^{1/2}q^s - \mathrm{e}^{-\partial_s}\sqrt{(1-q^{s+1})(1-\mu q^s)}\right] \end{aligned} \tag{9}$$

where $\partial_s \equiv \mathrm{d}/\mathrm{d}s$, $\mathrm{e}^{\alpha\partial_s} f(s) = f(s+\alpha)$. These operators are adjoint, $(a^+\psi, \chi) = (\psi, a\chi)$, with respect to the scalar product (8). They satisfy the q-commutation rule

$$aa^+ - qa^+a = 1 \tag{10}$$

and act on the q-wavefunctions defined in (7) by

$$a\psi_n = e_n^{1/2}\psi_{n-1} \qquad a^+\psi_n = e_{n+1}^{1/2}\psi_{n+1} \tag{11}$$

where

$$e_n = \frac{1-q^n}{1-q}\,.$$

In this model of the q-oscillator, equations (11) are equivalent to the following difference-differentiation formulae:

$$\mu q^s \Delta v_n^{\mu}(x;q) = (q^n - 1)\, v_{n-1}^{\mu}(x;q) \qquad q^s \nabla[\rho(s)\, v_n^{\mu}(x;q)] = \rho(s)\, v_{n+1}^{\mu}(x;q)$$

respectively. In view of (6) the functions $\psi_n(s)$ converge in the limit $q \to 1^-$ to the wavefunctions of the discrete model of the linear harmonic oscillator considered in [16].

The *q-Hamiltonian* $H = a^+a$ acts on the wavefunctions (7) as

$$H\psi_n = e_n\psi_n \tag{12}$$

and has the following explicit form:

$$H = (1-q)^{-1}\Big[\mu q^{2s} + (1-q^s)(1-\mu q^{s-1}) - \mu^{1/2}q^s\sqrt{(1-q^{s+1})(1-\mu q^s)}\,\mathrm{e}^{\partial_s} - \mu^{1/2}q^{s-1}\sqrt{(1-q^s)(1-\mu q^{s-1}}\,\mathrm{e}^{-\partial_s}\Big]. \tag{13}$$

By factorizing the Hamiltonian (or the difference equation for the Al-Salam and Carlitz polynomials) we arrive at the explicit form (9) for the q-boson operators.

Since $a^+a = H$, the relation (10) can be written in the equivalent form

$$[a, a^+] = 1 - (1-q)H \equiv q^N. \tag{14}$$

The operator

$$N = \frac{1}{\log q}\log[1-(1-q)H] \tag{15}$$

can be considered as the *number operator*, since

$$[a, N] = a \qquad [N, a^+] = a^+. \tag{16}$$

From these relations one can obtain the equations (11) and the spectrum (12) of the q-Hamiltonian in abstract form. The q-wavefunctions are

$$\psi_n(s) = c_n\left(a^+\right)^n\psi_0(s) \qquad a\,\psi_0(s) = 0$$

where $c_n = (e_n!)^{-1/2}$ and $e_n! = e_1 e_2 \ldots e_n$.

For the model of the q-oscillator under discussion we can construct explicitly *q-coherent states* and an analogue of the Fourier transformation. For the coherent states $|\,\alpha\rangle$ defined by

$$a\,|\,\alpha\rangle = \alpha\,|\,\alpha\rangle \qquad \langle\alpha\,|\,\alpha\rangle = 1$$
$$|\,\alpha\rangle = f_\alpha\sum_{n=0}^{\infty}\frac{\alpha^n\psi_n(s)}{(e_n!)^{1/2}} \tag{17}$$
$$f_\alpha = \left((1-q)\,|\,\alpha|^2; q\right)_\infty^{1/2} \qquad (1-q)\,|\,\alpha|^2 < 1$$

we can write

$$|\,\alpha\rangle = f_\alpha\left(\rho q^{-s}\right)^{1/2}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\,v_n^\mu(x;q),\ t = \alpha\mu^{1/2}(1-q)^{1/2}. \tag{18}$$

With the aid of the generating function [11]

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\,v_n^{\mu}(x;q)=\frac{(qtx/\mu;q)_\infty}{(t,qt/\mu;q)_\infty}\qquad |t|<1\qquad |qt/\mu|<1 \tag{19}$$

we arrive at the following *explicit form* for the q-coherent states:

$$|\,\alpha\rangle = f_\alpha\left(\rho q^{-s}\right)^{1/2}\frac{\left(\alpha(1-q)^{1/2}\mu^{-1/2}q^{1-s};q\right)_\infty}{\left(\alpha(1-q)^{1/2}\mu^{1/2},\alpha q(1-q)^{1/2}\mu^{-1/2};q\right)_\infty} \tag{20}$$

where $\rho=(q;q)_\infty^{-1}(q^{s+1},\mu q^s;q)_\infty\mu^s q^{s^2}$. These coherent states are not orthogonal:

$$\langle\alpha\mid\beta\rangle=\frac{\left((1-q)\mid\alpha\mid^2,\ (1-q)\mid\beta\mid^2;q\right)_\infty^{1/2}}{((1-q)\alpha^*\beta\,;q)_\infty}$$

where $*$ denotes the complex conjugate.

To define an analogue of the *Fourier transform* we can consider, following Wiener's approach to the classical Fourier transform [17] (see also [18, 19]), the kernel of the form

$$\begin{aligned}K_t(s,p)&=\sum_{n=0}^{\infty}t^n\psi_n(s)\,\psi_n(p)\\&=\left(\rho(s)\rho(p)q^{-s-p}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(\mu t)^n}{(q;q)_n}\,v_n^{\mu}(q^{-s};q)\,v_n^{\mu}(q^{-p};q)\,.\end{aligned} \tag{21}$$

The series can be summed with the aid of the bilinear generating function by Al-Salam and Carlitz [11]:

$$\begin{aligned}\sum_{n=0}^{\infty}v_n^{\mu_1}(x;q)\,v_n^{\mu_2}(y;q)\,\frac{t^n}{(q;q)_n}&=\frac{(qtx/\mu_1,qty/\mu_2;q)_\infty}{(t,qt/\mu_1,qt/\mu_2;q)_\infty}\\&\quad\times{}_3\varphi_2\left(\begin{matrix}x,y,t\\ qtx/\mu_1,qty/\mu_2\end{matrix};q,\frac{q^2t}{\mu_1\mu_2}\right).\end{aligned} \tag{22}$$

The answer is

$$K_t(s,p)=\left(\rho(s)\rho(p)q^{-s-p}\right)^{1/2}\frac{(tq^{1-s},tq^{1-p};q)_\infty}{(qt,qt,\mu t;q)_\infty}\,{}_3\varphi_2\left(\begin{matrix}q^{-s},q^{-p},\mu t\\ tq^{1-s},tq^{1-p}\end{matrix};q,\frac{q^2t}{\mu}\right). \tag{23}$$

The q-wavefunctions (7) are eigenfunctions of the 'discrete q-Fourier transform'

$$\mathrm{i}^m\psi_m(s)=\sum_{p=0}^{\infty}K_i(s,p)\,\psi_m(p)\,. \tag{24}$$

The orthogonality relation of the kernel

$$\sum_{p=0}^{\infty}K_i(s,p)\,K_i^*(s',p)=\delta_{ss'} \tag{25}$$

implies the orthogonality of the rational functions (23) and results in an inversion formula for the transformation:

$$\psi(s) = \sum_{p=0}^{\infty} K_i(s, p)\,\varphi(p) = F_q[\varphi](s)\,. \tag{26}$$

The kernel (23) of this q-Fourier transform is an eigenfunction of the following 'momentum' and 'position' operators:

$$\begin{aligned} P &= (\mu+q)\,q^{-1-N} + \mathrm{i}\sqrt{\mu(1-q)}\,\big(aq^{-N} - q^{-N}a^{+}\big) \\ Q &= (\mu+q)\,q^{-1-N} - \sqrt{\mu(1-q)}\,\big(aq^{-N} + q^{-N}a^{+}\big) \end{aligned} \tag{27}$$

namely

$$Q_s K_i(s, p) = q^{-s} K_i(s, p) \qquad P_s K_i(s, p) = q^{-p} K_i(s, p)\,. \tag{28}$$

Equations

$$P_s F_q[\varphi](s) = F_q[q^{-p}\varphi](s) \qquad F_q^{-1}[P_p\,\varphi](s) = q^{-s} F_q^{-1}[\varphi](s) \tag{29}$$

are analogues of the well known properties of the classical Fourier transform (cf [19]). In view of (6) in the limit $q \to 1^-$ we get one of the 'discrete Fourier transforms' considered in [18].

Similarly, with the aid of the bilinear generating function (22) and the orthogonality property of the Wall polynomials [14], which are dual to the polynomials (5), one can obtain the *biorthogonality relation*

$$\sum_{s=0}^{\infty} u_m(s)\, v_n(s)\, \rho(s) q^{-s} = d_n^2\, \delta_{mn} \tag{30}$$

with

$$\rho(s) = \frac{(\mu_2/t_1, \mu_2/t_2\,; q)_s}{(q, \mu_2; q)_s}\, \mu_1^s q^s$$

and

$$d_n^2 = \frac{(t_1, t_2; q)_\infty}{(\mu_1, \mu_2; q)_\infty} \frac{(q, \mu_1; q)_n}{(\mu_1/t_1, \mu_1/t_2; q)_n}\, \mu_2^{-n}$$

for the ${}_3\varphi_2$-*rational functions* of the form

$$\begin{aligned} u_m(s) &= {}_3\varphi_2 \left(\begin{matrix} q^{-m}, q^{-s}, t_1 \\ \dfrac{t_1}{\mu_1} q^{1-m}, \dfrac{t_1}{\mu_2} q^{1-s}; q, \dfrac{q^2}{t_2} \end{matrix} \right) \\ v_n(s) &= u_n(s)|_{t_1 \leftrightarrow t_2} \qquad t_1 t_2 = \mu_1 \mu_2\,. \end{aligned} \tag{31}$$

These functions are self-dual. They belong to classical biorthogonal rational functions [20, 21].

We have considered here the explicit form of q-boson operators which satisfy the commutation rule (10) when $0 < q < 1$. The case $q > 1$ is also interesting. It leads to another family of the Al-Salam and Carlitz polynomials [22].

For the models of the q-oscillator under discussion one can readily construct dynamical symmetry group $SU_q(1, 1)$ [4] and write explicitly irreducible representations $|j, m\rangle_q = \psi_{j+m}(s)\, \psi_{j-m}(s')$ of the group $SU_q(2)$ [1, 2].

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